Week 5 - Wednesday

# COMP 2230

# Last time

- Exam 1 post mortem
- Induction example

# Questions?

# Assignment 2

# Logical warmup

- At noon the hour and minute hands of a clock sit on top of each other perfectly.
- What's the next **exact** time they will again be on top of each other?

# Induction Examples

# Proof by mathematical induction

- To prove a statement of the following form:
  - $\forall n \in \mathbb{Z}$, where $n \geq a$, property $P(n)$ is true
- Use the following steps:
  1. Basis Step: Show that the property is true for $P(a)$
  2. Induction Step:
     - Suppose that the property is true for some $n = k$, where $k \in \mathbb{Z}, k \geq a$
     - Now, show that, with that assumption, the property is also true for $k + 1$

# Divisibility

- Prove that, for all integers $n \geq 1, 2^{2n} - 1$ is divisible by 3
- **Hint:** Use induction

# Inequality

- Prove that, for all integers $n \geq 3$, $2n + 1 < 2^n$
- **Hint:** Use induction

# Fibonacci

- The Fibonacci sequence is 1, 1, 2, 3, 5, 8, ...
- We can define it (recursively) as follows:
  - $F_0 = F_1 = 1$
  - $F_n = F_{n-1} + F_{n-2}, n \geq 2$
- Prove that, for all integers $n \geq 1$, $F_{4n-1}$ is divisible by 3
- **Hint:** Use induction

# Strong Induction

# Strong induction

- There are situations where we need more than the fact that the $k^{\text{th}}$ element maintains some property to prove the $k + 1^{\text{st}}$ element has the same property
- Strong induction allows us to use the $k^{\text{th}}$ element, the $k - 1^{\text{st}}$, element, the $k - 2^{\text{nd}}$ element, and so on
- This is usually most helpful when the subterms you are doing induction on are of unknown size

# Proof by strong induction

- To prove a statement of the following form:
  - $\forall n \in \mathbb{Z}$, where $n \geq a$, property $P(n)$ is true
- Use the following steps:
  1. Basis Step: Show that the property is true for $P(a), P(a+1), \ldots P(b-1), P(b)$, where $a \leq b, b \in \mathbb{Z}$
  2. Induction Step:
     - Suppose that the property is true for some $a \leq i < k$, where $k \in \mathbb{Z}$ and $k > b$
     - Now, show that, with that assumption, the property is also true for $k$

# Example

**Theorem:** For all integers $n \geq 2$, $n$ is divisible by a prime
**Proof:**

- **Basis step:** $(n = 2)$ The property is true for $n = 2$ because 2 is divisible by 2
- **Induction step:** Assume that all numbers $i$ where $2 \leq i < k$ are divisible by a prime, where $k \in \mathbb{Z}$
- **Case 1:** $k$ is prime
  - If $k$ is prime, $k = k \cdot 1$, therefore $k$ is divisible by a prime, namely itself
- **Case 2**: $k$ is composite
  - If $k$ is composite, $k = a \cdot b$, where $a, b \in \mathbb{Z}$ and $2 \leq a < k$ and $2 \leq b < k$
  - By the induction hypothesis, $a$ is divisible by some prime $p$
  - Thus, $k = p \cdot c \cdot b = p \cdot d$ and $k$ is divisible by prime $p$
- Since we have shown the basis step and induction step of strong mathematical induction, the claim is true. ■

# Example

**Theorem:** It takes exactly $n - 1$ steps to assemble a jigsaw puzzle with $n$ pieces

**Proof:**

- **Basis step:** $(n = 1)$ A puzzle with 1 piece takes $0$ steps to put together
- **Induction step:** Assume it takes $i - 1$ steps to put together puzzles of size $i$ where $1 \leq i < k$, where $k \in \mathbb{Z}$
  - The last step in a puzzle of size $k$ is putting together a subpuzzle of size $j$ and a subpuzzle of size $k - j$ where $j \in \mathbb{Z}$ and $1 \leq j < k$ and $1 \leq k - j < k$
  - By the induction hypothesis, it took $j - 1$ steps to put together one subpuzzle and $k - j - 1$ steps to put together the other
  - Thus, the total number of steps is $(j - 1) + (k - j - 1) + 1 = k - 1$ steps
- Since we have shown the basis step and induction step of strong mathematical induction, the claim is true. ■

# Well-Ordering Principle for Integers

- It turns out that the concept of truth through mathematical induction is equivalent to another principle
- Well-Ordering Principle for the Integers:
  - Let $S$ be a set containing one or more integers all of which are greater than some fixed integer. Then $S$ has a least element.

# Upcoming

# Next time…

- Second order linear homogeneous recurrence relations
- General recursion

# Reminders

- Keep working on Assignment 2
- Read 5.6, 5.7, 5.8, and 5.9